# Three-to-One Resonances Near the Equilateral Libration Points

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**The nonlinear stability of the triangular points ($L_4$ and $L_5$) in the restricted problem of three bodies is investigated using second-order expansions. The analysis is carried out for all values of $\mu$ near $\mu_3$ using Lie transforms and the method of multiple scales. The results of the two methods are in full agreement with each other. Equations are derived for the time variation of the amplitudes $A_1$ and $A_2$ and the phases $\beta_1$ and $\beta_2$ of the two modes of oscillation. For each value of $\mu$ near $\mu_3$, the family of long period orbits is given by a curve $\Gamma$ in the $|A_2| - |A_1|$ plane. At $\mu_3$, $\Gamma = \Gamma_3$ consists of the straightline, $A_2 = 0.181\,|A_1|$ and all long period orbits are unstable. For $\mu = \mu_3 + \epsilon$ with $\epsilon$ a small positive number, $\Gamma$ consists of a smooth branch that intersects the $|A_1|$ axis at the origin and tends to $\Gamma_3$ as $\epsilon \to 0$. For $\mu = \mu_3 - \epsilon$, on the other hand, $\Gamma$ consists of two branches intersecting the $|A_1|$ axis at a point different from the origin. As $\epsilon \to 0$, the branch near the origin collapses while the other branch tends to $\Gamma_3$. Except at $\mu_3$, a cut-off value for $|A_1|$ exists above which long period orbits are unstable and below which they are stable.**

## 1. Introduction

IN the two-dimensional restricted problem of three bodies, two of the three bodies have finite masses (primaries) and the third has a negligible mass. The primaries are represented by point masses revolving in circular orbits about their common mass center uninfluenced by the third body. The motion of this third body is governed by the primaries, and remains in their plane of motion.

In a coordinate system rotating with the primaries, Lagrange (see Refs. 1 and 2) showed that there are five equilibrium points; three are colinear with the primaries ($L_1$, $L_2$, and $L_3$) and the other two are at the apexes of two equilateral triangles ($L_4$ and $L_5$) whose other vertices are at the primaries. A linear analysis of the motion around these points shows that the first three are unstable for all $\mu(\mu = m_2/m_1 + m_2$, $m_1$ and $m_2$ are the masses of the two primaries). On the other hand. the linear analysis shows that $L_4$ and $L_5$ unstable if $\mu > \tilde{\mu} = \frac{1}{2}[1 - 69^{1/2}/9]$. Below $\tilde{\mu}$, the motion is stable and can be represented as the superposition of two oscillatory modes with circular frequencies $\omega_1$ and $\omega_2$. The value $\tilde{\mu}$ is referred to as the critical mass ratio because it separates stable from unstable triangular points.

Leontovic[3] proved that $L_4$ and $L_5$ are stable in the sense of Kolmogorov-Arnold-Moser for all $\mu < \tilde{\mu}$ except on a set of measure zero. Deprit and Deprit-Bartholome[4] proved that the exceptional set contains at most four values of $\mu$ including $\mu_2 = 0.024295$ and $\mu_3 = 0.013516$. Markeev[5] proved instability of $L_4$ and $L_5$ at $\mu_2$ and $\mu_3$. The purpose of the present paper is to study the stability of motions near $L_4$ and $L_5$ for all values of $\mu$ near $\mu_3$ using Lie transforms[6,7] and the method of multiple scales.[8,9]

## 2. Problem Formulation

In the restricted problem of three bodies, two of these bodies have finite masses ($m_1$ and $m_2$) and revolve around one another in circular orbits with frequency $n$ ($n = [G(m_1 + m_2)/D^3]^{1/2}$, $G$ is the Newtonian gravitational constant and $D$ is the mean distance between the finite masses). The third body has an infinitesimal mass and moves in the field of the finite masses without affecting their motion. To analyze the motion of the infinitesimal mass, a Cartesian coordinate system rotating with the finite masses and centered at their center of mass is introduced such that they lie along the $\bar{x}$ axis. If distances and time are made dimensionless using the characteristic length $D$ and characteristic frequency $n$, the equations of motion of the infinitesimal mass are[2]

$$\ddot{\bar{x}} - 2\dot{\bar{y}} = -\partial V/\partial \bar{x} \tag{1a}$$

$$\ddot{\bar{y}} + 2\dot{\bar{x}} = -\partial V/\partial \bar{y} \tag{1b}$$

$$-V = \tfrac{1}{2}(\bar{x}^2 + \bar{y}^2) + (1 - \mu)/r_1 + \mu/r_2 \tag{1c}$$

$$r_1{}^2 = (\bar{x} - \bar{x}_1)^2 + \bar{y}^2, \quad r_2{}^2 = (\bar{x} - \bar{x}_2)^2 + \bar{y}^2 \tag{1d}$$

where dots denote differentiation with respect to time, $\bar{x}_1 = -\mu$ and $\bar{x}_2 = 1 - \mu$. There are five positions of equilibrium for the infinitesimal mass. They are given by the equation[1,2]

$$\nabla V = 0 \tag{2}$$

Three of these equilibrium points ($L_1$, $L_2$, and $L_3$) lie along the $\bar{x}$ axis and are unstable. The other two points ($L_4$ and $L_5$) form equilateral triangles in the $\bar{x}$-$\bar{y}$ plane with the two finite mass.

To investigate the stability of $L_4$ (the same results hold for $L_5$), the body is displaced to

$$\bar{x} = \tfrac{1}{2}(1 - 2\mu) + x, \quad \bar{y} = 3^{1/2}/2 + y \tag{3}$$

The right-hand sides of Eqs. (1a) and (1b) are expanded in Taylor series keeping up to third-order terms in $x$ and $y$. The resulting equations are

$$\ddot{x} - 2\dot{y} - \tfrac{3}{4}x - \eta y = f_{11} + f_{12}, f_{11} = -H_{1x}, f_{12} = -H_{2x} \tag{4a}$$

$$\ddot{y} + 2\dot{x} - \eta x - \tfrac{9}{4}y = f_{21} + f_{22}, f_{21} = -H_{1y}, f_{22} = -H_{2y} \tag{4b}$$

$$H_1 = 3(3^{1/2}/16)y(x^2 + y^2) + (3^{1/2}/36)\eta x(33y^2 - 7x^2) \tag{4c}$$

$$H_2 = \tfrac{37}{128}x^4 - \tfrac{123}{64}x^2y^2 - \tfrac{3}{128}y^4 + \tfrac{5}{24}\eta xy(5x^2 - 9y^2) \tag{4d}$$

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where $\eta = 3(3)^{1/2}(1 - 2\mu)/4$, and the subscripts $x$ and $y$ denote partial derivatives. These equations are appropriate for the application of the method of multiple scales, but the corresponding Hamiltonian is needed for application of Lie transforms. If the conjugate momenta $p_x$ and $p_y$ are defined by

$$p_x = \dot{x} - y, \quad p_y = \dot{y} + x \tag{5a}$$

then the Hamiltonian corresponding to Eq. (4) is

$$H = H_o + H_1 + H_2 \tag{5b}$$

where

$$H_0 = \tfrac{1}{2}(p_x^2 - p_y^2) + (yp_x - xp_y) + \tfrac{1}{8}(x^2 - 5y^2) - \eta xy \tag{5c}$$

The equations of motion (4) can be written in terms of $H$ as

$$\dot{x} = H_{p_x}, \quad \dot{p}_x = -H_x \tag{6a}$$

$$\dot{y} = H_{p_y}, \quad \dot{p}_y = -H_y \tag{6b}$$

If the nonlinear terms ($H_1$ and $H_2$) are neglected, the solution of the resulting equations (corresponding to $H_o$) can be obtained by assuming that $x$ and $y$ are proportional to $\exp(i\omega t)$ where

$$\omega^4 - \omega^2 + \tfrac{27}{16} - \eta^2 = 0 \tag{7}$$

The four roots of this equation depend on the value of $\mu$. If $0 \leq \mu < \tilde{\mu} = \frac{1}{2}(1 - 69^{1/2}/9)$, the four roots ($\pm\omega_1$ and $\pm\omega_2$) are real, and hence the equilibrium of a particle at $L_4$ is stable. On the other hand, if $\mu \geq \tilde{\mu}$, the equilibrium is unstable. In the stable case, the solution can be written in the form[10,11]

$$\begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} = J \begin{bmatrix} (A_1/\omega_1)\sin B_1 \\ (A_2/\omega_2)\sin B_2 \\ A_1 \cos B_1 \\ -A_2 \cos B_2 \end{bmatrix} \tag{8a}$$

where

$$A_i = (2\omega_i\alpha_i)^{1/2},\ B_1 = \omega_1(t + \beta_1),\ B_2 = \omega_2(t - \beta_2) \tag{8b}$$

$$J = \begin{bmatrix} 0 & 0 & (k_1/\omega_1)(\omega_1^2 + \frac{9}{4}) & -(k_2/\omega_2)(\omega_2^2 + \frac{9}{4}) \\ -2k_1\omega_1 & -2k_2\omega_2 & -(k_1/\omega_1)\eta & (k_2/\omega_2)\eta \\ -k_1\omega_1(\omega_1^2 + \frac{1}{4}) & -k_2\omega_2(\omega_2^2 + \frac{1}{4}) & (k_1/\omega_1)\eta & -(k_2/\omega_2)\eta \\ k_1\omega_1\eta & k_2\omega_2\eta & (k_1/\omega_1)(\frac{9}{4} - \omega_1^2) & -(k_2/\omega_2)(\frac{9}{4} - \omega_2^2) \end{bmatrix} \tag{8c}$$

$$k_i = |11\omega_i^2/2 + 2\eta^2 - 45/8|^{-1/2} \tag{8d}$$

Here, $\beta_1$, $\beta_2$, $\alpha_1$, and $\alpha_2$ are the constants of integration. Substitution of Eq. (8a) into $H_o$ leads to

$$H_o = \omega_1\alpha_1 - \omega_2\alpha_2 \tag{9}$$

Hence, $B_1$ and $-B_2$ are the coordinates and the $\alpha_i$'s are the conjugate momenta of a canonical set.

Since nonlinear terms are neglected, the previous solution is valid only for very small amplitudes $A_1$ and $A_2$. Any attempt to extend the linear solution to larger amplitudes by a straightforward perturbation or by iteration will fail near $\mu_2$, $\mu_3$, and $\mu_4$ because of the appearance of secular terms or small divisors. In these cases, alternative methods such as the method of multiple scales[8,9] and Hamiltonian methods with suitable canonical transformations[6,7,12] should be used. In this paper, the method of multiple scales and the recurrence formulas of Refs. 6 and 7 are used to study the stability of $L_4$ and $L_5$ when $\omega_1 \approx 3\omega_2$ {i.e., $\mu \approx \mu_3 = \frac{1}{2}[1 - (71/75^{1/2}] = 0.013516$}.

## 3. Method of Multiple Scales

A perturbation solution for finite but small amplitudes is obtained in this section using the method of multiple scales. According to this method, $x$ and $y$ are assumed to be functions of two independent time scales, a fast time, $T_0 = t$, and a slow time, $T_2 = \epsilon^2 t$, where $\epsilon$ is a small parameter of the order of the amplitude of the first-order solution. To third-order terms in $\epsilon$, $x$ and $y$ are assumed to possess the following uniformly valid expansions for all times:

$$x(t;\epsilon) = \sum_{n=1}^{3} \epsilon^n x_n(T_0,T_2) + 0(\epsilon^4) \tag{10}$$

$$y(t;\epsilon) = \sum_{n=1}^{3} \epsilon^n y_n(T_0,T_2) + 0(\epsilon^4) \tag{11}$$

The time derivative becomes

$$d/dt = D_0 + \epsilon^2 D_2 + \ldots, \quad D_n = \partial/\partial T_n \tag{12}$$

Substituting Eqs. (10) to (12) into Eqs. (4a) and (4b) expanding and equating coefficients of equal powers of $\epsilon$ on both sides lead to

**Order $\epsilon$**

$$M_1(x_1,y_1) = D_0^2 x_1 - 2D_0 y_1 - \tfrac{3}{4}x_1 - \eta y_1 = 0 \tag{13}$$

$$M_2(x_1,y_1) = D_0^2 y_1 + 2D_0 x_1 - \eta x_1 - \tfrac{9}{4}y_1 = 0 \tag{14}$$

**Order $\epsilon^2$**

$$M_1(x_2,y_2) = f_{11}(x_1,y_1) \tag{15}$$

$$M_2(x_1,y_2) = f_{21}(x_1,y_1) \tag{16}$$

**Order $\epsilon^3$**

$$M_1(x_3,y_3) = f_{12}(x_1,y_1) + x_2 f_{11x}(x_1,y_1) + y_2 f_{11y}(x_1,y_1) - 2D_0D_2x_1 + 2D_2y_1 \tag{17}$$

$$M_2(x_3,y_3) = f_{22}(x_1,y_1) + x_2 f_{21x}(x_1,y_1) + y_2 f_{21y}(x_1,y_1) - 2D_0D_2y_1 - 2D_2x_1 \tag{18}$$

where the subscripts $x$ and $y$ denote partial derivatives.

The general solution to Eqs. (13) and (14) is given by Eq. (8) if $A_i$ and $\beta_i$ are considered functions of the slow time $T_2$ rather than being constants. These arbitrary functions will be determined in the course of analysis. If $A_i$ is small, then $\epsilon$ can be used only to keep track of the ordering, and will be set equal to unity in the final solution. Substitution of Eq. (8) into Eqs. (15) and (16) yields

$$\begin{aligned} M_i(x_2,y_2) = {} & a_{i0}A_1^2 + a_{i5}A_2^2 + a_{i1}A_1^2 \cos 2B_1 + b_{i1}A_1^2 \sin 2B_1 + a_{i2}A_2^2 \cos 2B_2 + b_{i2}A_2^2 \sin 2B_2 + \\ & a_{i3}A_1A_2 \cos(B_2 + B_1) + b_{i3}A_1A_2 \sin(B_2 + B_1) + a_{i4}A_1A_2 \cos(B_1 - B_2) + b_{i4}A_1A_2 \sin(B_1 - B_2), \quad i = 1,2 \end{aligned} \tag{19}$$

where the $a$'s and $b$'s are given in Appendix A. The solution of Eq. (19) is

$$\begin{aligned} x_2^{(i)} = {} & c_{i0}A_1^2 + c_{i5}A_2^2 + c_{i1}A_1^2 \cos 2B_1 + s_{i1}A_1^2 \sin 2B_1 + c_{i2}A_2^2 \cos 2B_2 + s_{i2}A_2^2 \sin 2B_2 + \\ & c_{i3}A_1A_2 \cos(B_1 + B_2) + s_{i3}A_1A_2 \sin(B_1 + B_2) + c_{i4}A_1A_2 \cos(B_1 - B_2) + s_{i4}A_1A_2 \sin(B_1 - B_2) \end{aligned} \tag{20}$$

where $x_2^{(1)} = x_2$ and $x_2^{(2)} = y_2$, and the $c$'s and $s$'s are given in Appendix B.

In order to avoid lengthy algebraic expressions, the rest of the analysis is carried out for three specific cases; namely, $\mu = \mu_3 = \frac{1}{2}[1 - (71/75)^{1/2}] \approx 0.013516$, $\mu = 0.0015$, and $\mu = 0.0125$. The analysis is carried out in detail for the former case, while the results without the analysis are presented for the other values of $\mu$ because the analysis is independent of the value of $\mu$. These results are used to obtain the solution for all values of $0.0125 \leq \mu \leq 0.015$ in the form of parabolic functions of $(\mu - \mu_3)$.

### A. Case of $\mu_3$

Setting $\mu = \mu_3$ in Eq. (8) and Eq. (20), and substituting the results into Eq. (17) and Eq. (18) yields

$$M_1(x_3,y_3) = \sum_{i=1}^{2} (P_{1i} \sin B_i + Q_{1i} \cos B_i) + R_1 \quad (21)$$

$$M_2(x_3,y_3) = \sum_{i=1}^{2} (P_{2i} \sin B_i + Q_{2i} \cos B_i) + R_2 \quad (22)$$

where the $P$'s and $Q$'s are functions of the $A$'s and $B$'s and their derivatives; they are given in Appendix C. The $R$'s do not contain terms of the form $S_{1i} \cos B_i$ and $S_{2i} \sin B_i$ where $S_{ji}$ are functions of $T_2$ or $\bar{\gamma}$.

The particular solution of Eqs. (21) and (22) contains secular terms of the form $T_0 \cos B_i$ and $T_0 \sin B_i$ which make $x_3$ and $y_3$ become unbounded as $T_0 \to \infty$, i.e., $t \to \infty$. Hence, $x_3$ and $y_3$ will dominate the lower-order terms, and the expansion will break down for large $t$ unless the secular terms are eliminated. The conditions which must be satisfied for there to be no secular terms in $x_3$ and $y_3$ are

$$-2\omega_i Q_{2i} + \eta P_{2i} = (\omega_i^2 + \tfrac{9}{4})P_{1i} \quad (23)$$

$$2\omega_i Q_{2i} + \eta P_{2i} = (\omega_1^2 + \tfrac{9}{4})Q_{1i} \quad (24)$$

$i = 1$ and 2. Using the expressions for the $P$'s and $Q$'s from Appendix C, and solving the resulting equations for $A_i'$ and $\beta_i'$ lead to

$$A_1' = -6.061 \sin\gamma A_2^3 \quad (25)$$

$$A_2' = -6.061 \sin\gamma A_1 A_2^2 \quad (26)$$

$$A_1\beta_1' = 0.1537 A_1^3 - 3.626 A_1 A_2^2 - 6.389 \cos\gamma A_2^3 \quad (27)$$

$$\beta_2' = -3.626 A_1^2 - 19.17 \cos\gamma A_1 A_2 + 2.489 A_2^2 \quad (28)$$

where

$$\gamma = \omega_1(\beta_1 + \beta_2) - 17° \quad (29)$$

Elimination of $\beta_1$ and $\beta_2$ from Eqs. (27–29) gives the following equation for $\gamma$:

$$A_1\gamma' = -3.294 A_1^3 - 18.18 \cos\gamma A_1^2 A_2 - 1.079 A_1 A_2^2 - 6.061 \cos\gamma A_2^3 \quad (30)$$

#### *Integrals of the Motion*

The elimination of $\gamma$ from Eqs. (25) and (26), and the solution of the resulting equation lead to

$$A_1^2 = A_2^2 + c_1 \quad (31)$$

where $c_1$ is an arbitrary constant. Letting $z = 6.137 \cos\gamma$, and differentiating with respect to $T_2$ yield

$$z' = -6.061 \sin\gamma \gamma' \quad (32)$$

From Eqs. (26) and (32) there results

$$dz/dA_2 = \gamma'/A_1 A_2^2 \quad (33)$$

The elimination of $\gamma'$ between Eqs. (30) and (33) gives

$$\frac{dz}{dA_2} + \left(\frac{3}{A_2} + \frac{A_2}{A_1}\right) z = -3.294 \frac{A_1}{A_2^2} - 1.079 \frac{1}{A_1} \quad (34)$$

Since $A_1^2 = A_2^2 + c_1$, Eq. (34) is a linear differential equation for $z$ in terms of $A_2$. It has the following integral

$$A_1 A_2^3 z + 0.8235 A_1^4 + 0.2697 A_2^4 = c_2 \quad (35)$$

where $c_2$ is an arbitrary constant. This latter integral corresponds to the Hamiltonian as will be shown in the next section. In order to specify the motion completely, a third integral is needed which seems to be available by numerical integration only.

#### *Periodic Orbits and Their Stability*

Before considering long period orbits ($L_4{}^l$) let us consider short period orbits ($L_4{}^s$). Equations (25–28) admit stationary solutions corresponding to $L_4{}^s$ ($A_2 = 0$ and $A_1 = a =$ constant) with a period $\tau$ given by

$$\tau = 2\pi/\omega_1(1 + 0.1537a^2) \quad (36)$$

From Eq. (30),

$$\gamma = -3.294a^2 T_2 + \gamma_0 \quad (37)$$

where $\gamma_0$ is a constant. If a small disturbance $\delta A_2$ and $\delta A_1$ is introduced at $t = 0$, then from Eqs. (25) and (26)

$$1/A_2 = 1/\delta A_2 - (1.840/a)(\cos\gamma - \cos\gamma_0) \quad (38)$$

Then, $A_1$ can be obtained from Eq. (31). Equation (38) shows that any small disturbance applied to an $L_4{}^s$ results in an aperiodic, though stable, motion. Thus, in this sense, $L_4{}^s$ are unstable.

Long period orbits correspond to stationary solutions of Eqs. (25, 26, and 30). They are given by the following equations:

$$\sin\gamma_0 = 0 \quad (39)$$

$$3.294 A_{10}^3 + 18.18 \cos\gamma_0 A_{10}^2 A_{20} + 1.079 A_{10} A_{20}^3 + 6.061 \cos\gamma_0 A_{20}^3 = 0 \quad (40)$$

whose solution is

$$\gamma_0 = n\pi, \quad n \text{ is an integer} \quad (41a)$$

$$A_{20}/A_{10} = -0.1811 \cos n\pi \quad (41b)$$

The period of any of these orbits is

$$\tau = 2\pi/\omega_2(1 - 0.0728 A_{10}^2) \quad (41c)$$

The stability of these solutions is determined by letting

$$A_1 = A_{10} + \Delta A_1 \exp(sT_2) \quad (42a)$$

$$A_2 = A_{20} + \Delta A_2 \exp(sT_2) \quad (42b)$$

$$\gamma = n\pi + \Delta\gamma \exp(sT_2) \quad (42c)$$

Substituting Eq. (42) into Eqs. (25, 26, and 30), expanding for small $\Delta A_1$, $\Delta A_2$, and $\Delta\gamma$, keeping linear terms only, and setting the determinant of the resulting linear equations to zero give

$$s^2 = 3.537 A_{10}^4 \quad (43)$$

Since $s^2$ is always positive, $L_4{}^l$ are unstable for $\mu = \mu_3$.

### B. General Case

Carrying out the analysis for the remaining two values of $\mu$ leads to the following equations

$$A_1' = -\omega_1 e_4 A_2^3 \sin\gamma \quad (44)$$

$$A_2' = -3\omega_2 e_4 A_1 A_2^2 \sin\gamma \quad (45)$$

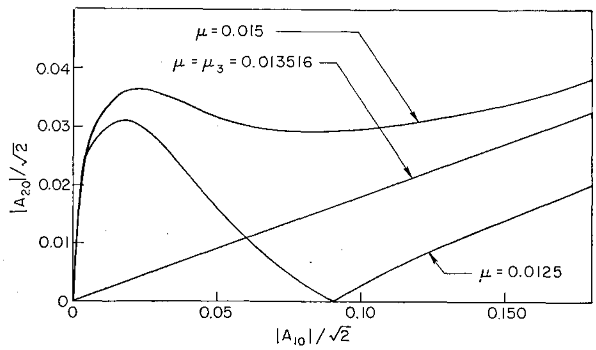


**Fig. 1 Family of long period orbits.**

$$A_1\beta_1' = 2e_1A_1^3 + e_2A_1A_2^2 - e_4A_2^3 \cos\gamma \quad (46)$$

$$\beta_2' = e_2A_1^2 + 2e_3A_2^2 - 3e_4A_1A_2 \cos\gamma \quad (47)$$

$$\gamma = \sigma T_2 + \omega_1\beta_1 + 3\omega_2\beta_2 + \phi,\ \sigma = \omega_1 - 3\omega_2 \quad (48)$$

where $e_1 = 0.06686$, $e_2 = -3.150$, $e_3 = 1.741$, $e_4 = 6.815$, $\tan\phi = 0.1531$ for $\mu = 0.0125$, and $e_1 = 0.09360$, $e_2 = -4.542$, $e_3 = 0.4590$, $e_4 = 8.565$, $\tan\phi = -1.1411$ for $\mu = 0.015$.

The numerical results for $\mu = \mu_3$, 0.0125, and 0.015 are used to express the $e$'s, $\tan\phi$, and the detuning $\sigma$ in the following parabolic functions:

$$e_1 = 0.07687 + 10.43\Delta + 569.6\Delta^2 \quad (49a)$$

$$e_2 = -3.626 - 529.3\Delta - 59580\Delta^2 \quad (49b)$$

$$e_3 = 1.244 - 505.5\Delta - 16020\Delta^2 \quad (49c)$$

$$e_4 = 6.389 - 694.2\Delta + 3.018 \times 10^6\Delta^2 \quad (49d)$$

$$\tan\phi = -0.3050 + 9333\Delta - 1.800 \times 10^5\Delta^2 \quad (49e)$$

$$\sigma = -21.65\Delta + 179.3\Delta^2,\ \Delta = \mu - \mu_3 \quad (49f)$$

Elimination of $\beta_1$ and $\beta_2$ from Eqs. (46–48) gives

$$A_1\gamma' = A_1\sigma + e_5A_1^3 - 9\omega_2e_4A_1^2A_2 \cos\gamma + e_6A_1A_2^2 - \omega_1e_4A_2^3 \cos\gamma \quad (50a)$$

$$e_5 = 2e_1\omega_1 + 3e_2\omega_2 \quad (50b)$$

$$e_6 = e_2\omega_1 + 6e_3\omega_2 \quad (50c)$$

### *Integrals of Motion*

Elimination of $\gamma$ between Eqs. (44) and (45) and solution of the resulting equation yield

$$3\omega_2A_1^2 = \omega_1A_2^2 + c_1 \quad (51)$$

where $c_1$ is an arbitrary constant. This integral does not provide any bound on $A_1$ and $A_2$ because they may grow individually as long as $3\omega_2A_1^2 - \omega_1A_2^2$ is constant. It should be mentioned that the present second-order theory ceases to be valid before $A_1$ and $A_2$ have grown to very large values, and higher-order terms need to be included in the analysis. As in the case of $\mu_3$, the following second-order integral can be obtained:

$$3\omega_2e_4A_1A_2^3 \cos\gamma = \frac{1}{2}\sigma A_2^2 + \frac{3\omega_2e_5}{4\omega_1}A_1^4 + \frac{1}{4}e_6A_2^4 + c_2 \quad (52)$$

where $c_2$ is another constant of integration. As in the $\mu = \mu_3$ case, one more integral is needed to complete the description of the motion; this integral also seems to be available by numerical integration only.

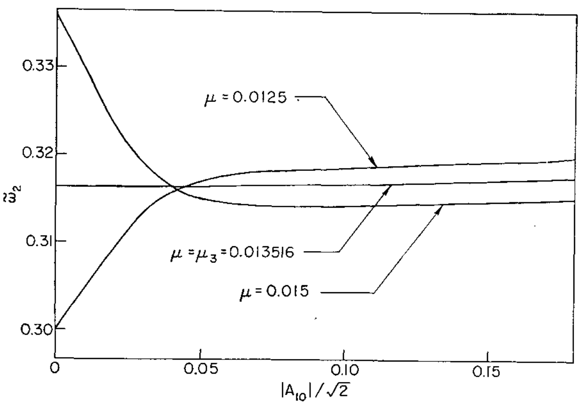


**Fig. 2 Variation of the frequency of long period orbits with amplitude of the first mode.**

### *Periodic Orbits and Their Stability*

Equations (44–47) admit stationary solutions corresponding to $L_4^s$ with $A_2 = 0$ and $A_1 = a$ whose period is

$$\tau = 2\pi/\omega_1(1 + 2e_1a^2) \quad (53)$$

These periodic orbits are unstable in the sense that if a disturbance $\delta A_2$ and $\delta A_1$ is introduced, then

$$\frac{1}{A_2} = \frac{1}{\delta A_2} + \frac{3\omega_2e_4a}{\sigma + e_5a^2}(\cos\gamma - \cos\gamma_0) \quad (54)$$

Thus, a slight disturbance in $A_2$ results in an aperiodic, though stable, motion.

Long period orbits correspond to stationary solutions of Eqs. (44, 45, and 50). They are solutions of

$$\sin\gamma_0 = 0 \text{ or } \gamma_0 = n\pi \quad (55a)$$

$$A_{10}\sigma + e_5A_{10}^3 - 9\omega_2e_4A_{10}^2A_{20} \cos\gamma_0 + e_6A_{10}A_{20}^2 - \omega_1e_4A_{20}^3 \cos\gamma_0 = 0 \quad (55b)$$

For a given value of $\mu$, Eq. (55) shows that $L_4^l$ are given by a third-order curve ($\Gamma$) in the $A_{10} - A_{20}$ plane. Figure 1 shows three of these curves in the $|A_{10}| - |A_{20}|$ plane for $\mu = 0.0125$, 0.013516, and 0.015. The behavior of these curves depends on the value of $\mu$. For values of $\mu \geq \mu_3$ such as 0.015, $\Gamma$ consists of one branch that intersects the $|A_{10}|$ axis at the origin only, and as $\mu \to \mu_3$, $\Gamma$ tends to $L_4^l$ at $\mu_3$. On the other hand, for values of $\mu < \mu_3$ (negative $\Delta$ and positive detuning $\sigma$), $\Gamma$ consists of two distinct branches which connect at a point $\tilde{A}_{10}$ on the $A_{10}$ axis where

$$\tilde{A}_{10} = 3.601(-\Delta)^{1/2}(1 + 107.7\Delta) \quad (56)$$

The variation of $\tilde{A}_{10}$ with $\Delta$ is shown in Fig. 2. As $\mu \to \mu_3$, the branch between the origin and $\tilde{A}_{10}$ collapses while the second branch tends to $L_4^l$ at $\mu_3$.

It should be mentioned here that our results are not valid outside the interval $\mu^{**} \leq \mu \leq \mu^*$. If $\mu > \mu^*$, $L_4^l$ terminates on a short period orbit travelled two times, while if $\mu < \mu^{**}$, $L_4^l$ terminates on a short period orbit travelled four times. Deprit and Price[13] have shown that $0.01226 < \mu^{**} < 0.012316$. Therefore, the analysis of $L_4$ for the Earth-moon system escapes the three-to-one model set up by Breakwell and Pringle[11] and Schechter,[14] and should be treated with a four-to-one model[15] because the Earth-moon $\mu$ is $0.01215 < \mu^{**}$.

These general periodic solutions ($L_4^l$) are described by a superposition of both normal modes (i.e., $A_{10} \neq 0$ and $A_{20} \neq 0$). Periodicity is achieved through adjustment of the frequencies of both modes via the nonlinear coupling which makes them exactly commensurable. The frequencies of the

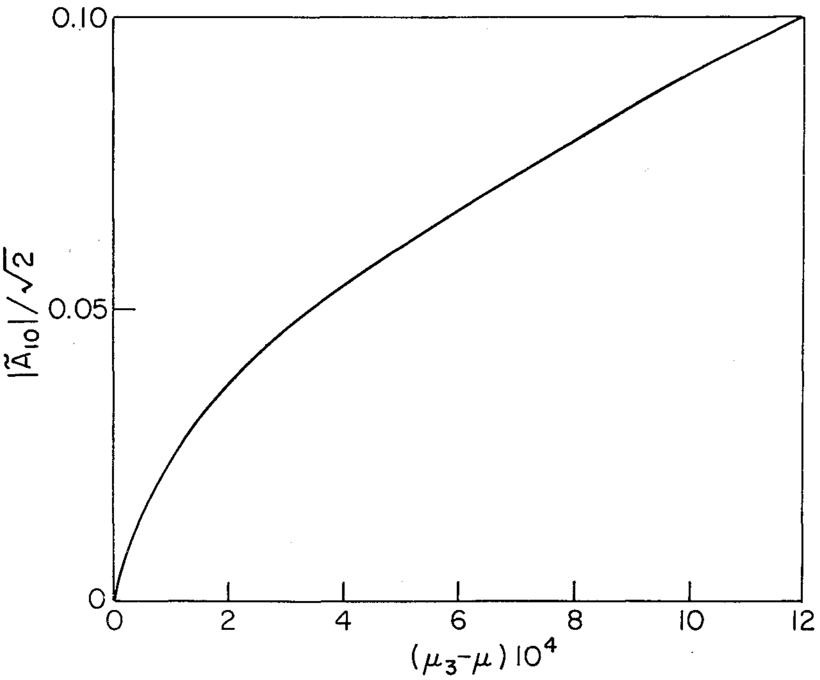


**Fig. 3 Dependence on $\Delta$ ($=\mu-\mu_3$) of the intersection of $\Gamma$ with the $|A_{10}|$ axis ($\Gamma$ is the family of long period orbits).**

two modes $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are given by

$$\tilde{\omega}_1 = \omega_1(1+\beta_1') = 3\omega_2(1-\beta_2') = 3\tilde{\omega}_2 \tag{57}$$

The variation of $\tilde{\omega}_2$ with $A_{10}$ is shown in Fig. 3 for $\mu$ = 0.013516, 0.015, and 0.0125. The figure shows that a slight nonlinearity adjusts $\omega_1$ to a value $\tilde{\omega}_1 = 3\omega_2$. As the amplitude increases, the frequencies of both modes get adjusted to become commensurable.

The stability of these periodic orbits can be obtained, as in the case of $\mu_3$, by determining the value of $s$ in Eqs. (42) to be

$$s^2 = -e_4A_{20}^2\cos\gamma_0(\omega_1A_{20}R_1/A_{10} + 3\omega_2R_2) \tag{58}$$

$$R_1 = \sigma + 3e_5A_{10}^2 - 18\omega_2e_4A_{10}A_{20}\cos\gamma_0 + e_6A_{20}^2 \tag{59a}$$

$$R_2 = -9\omega_2e_4A_{10}^2\cos\gamma_0 + 2e_6A_{10}A_{20} - 2\omega_1e_4A_{20}\cos\gamma_0 \tag{59b}$$

To determine the stability of a periodic orbit, the values of $A_{10}$, $A_{20}$, and $\gamma_0$ are substituted into Eq. (58). Then the orbit is stable or unstable according to whether $s^2$ is negative or positive. As shown earlier, if $\mu_3 = \mu$, then $s^2 > 0$ for all values of $A_{10}$ and $A_{20}$. If $\mu \neq \mu_3$, $s^2$ is negative for small $|A_{10}|$ and positive for large $|A_{10}|$. Evaluation of Eq. (58) on $\Gamma$ yields a cutoff value for $|A_{10}|$ above which long period orbits are unstable and below which they are stable. For stable orbits, $|A_{10}|$ must be less than or equal to $0.014(2)^{1/2}$ and $0.012(2)^{1/2}$ for $\mu$ = 0.015 and 0.0125, respectively.

For small amplitudes, long period orbits are ellipses, while for large amplitudes they become distorted ellipses due to the influence of the higher harmonics.

### *Stability of Nonperiodic Motions*

To determine the stability of nonperiodic solutions, let

$$\zeta = \omega_1A_2^2/|c_1| \tag{60a}$$

$$A_1^2 = |c_1|(\zeta \pm 1)/3\omega_2 \tag{60b}$$

Elimination of $\gamma$ from Eqs. (45) and (52), and using Eq. (60) give

$$(\omega_1/12\omega_2)(\zeta'/c_1e_4)^2 = F^2(\zeta) - G^2(\zeta) \tag{61a}$$

$$F(\zeta) = \pm[\zeta^3(\zeta\pm 1)]^{1/2} \tag{61b}$$

$$G(\zeta) = (\omega_1/3\omega_2e_4^2)^{1/2}[(e_5/12\omega_2)(\zeta\pm 1)^2 + (e_6/4\omega_1)\zeta^2 + \sigma\zeta/2|c_1|] + \text{constant} \tag{61c}$$

where the plus and minus signs in the parentheses correspond to positive and negative $c_1$, respectively. The constant in Eq. (61) is a function of $\mu$.

The functions $F(\zeta)$ and $G(\zeta)$ are shown schematically in Fig. 4. The reality of the motion requires that $F^2 \geq G^2$. The points where $G$ meet $F$ correspond to the vanishing of both $A_1'$ and $A_2'$. A curve such as $G_2$ which meets both branches of $F$, or meets one branch at two different points, corresponds to a bounded aperiodic motion. The amplitudes $A_1$ and $A_2$ of the two modes of motion correspond to points below the curve $\Gamma$ in Fig. 1. On the other hand, a curve such as $G_4$ which meets $F$ at one point only represents an unbounded motion, and the amplitudes $A_1$ and $A_2$ correspond to points above $\Gamma$. The points $P_1$ and $P_3$ where $G_1$ and $G_3$ touch $F$ represent equilibrium points (periodic orbits) because the relationship $F' = G'$ is equivalent to Eqs. (55); the locus of $P_1$ and $P_3$ is the curve $\Gamma$. A point such as $P_1$ corresponds to a stable periodic orbit, whereas $P_3$ correspond to an unstable periodic orbit.

## 4. Lie Transforms

The solution of the nonlinear problem corresponding to the Hamiltonian Eq. (5) can be obtained starting from the linear

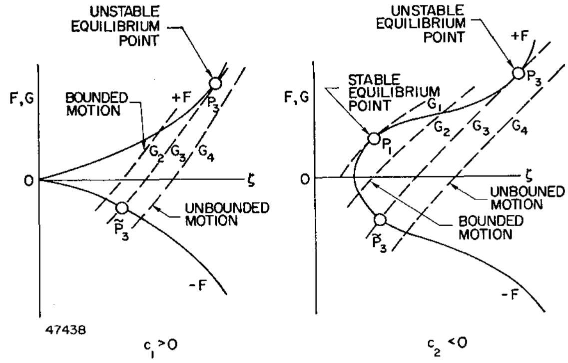


**Fig. 4 Topology of motion.**

solution using the method of variation of parameters. If $\alpha_i$ and $\beta_i$ are allowed to be functions of time rather than constants, then the form of the solution remains the same as the linear solution (8). If the linear solution is substituted into Eq. (5) then $\alpha_i$ and $\beta_i$ satisfy the following Hamilton's equations:

$$\dot{\alpha}_i = -\partial H/\partial B_i, \quad \dot{B}_i = (-1)^{i+1}\partial H/\partial \alpha_i \tag{62}$$

Here $H$ contains fast (short period) as well as slowly varying (long period) terms. Since the interest is in the over-all broad features of the motion, the long period behavior of $\alpha_i$ and $\beta_i$ need only be investigated. This can be accomplished by transforming from the canonical set $\alpha_i, B_i$ corresponding to $H$ to a new set $\alpha_i^*$, $B_i^*$ corresponding to a slowly varying Hamiltonian $K$. The transformation is accomplished by employing the recursive formulas of Refs. 6 and 7.

It is shown in Refs. 6 and 7 that given a Hamiltonian $H(x,X,t;\epsilon)$ in the form

$$H(x,X,t;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n(x,X,t) \tag{63}$$

then one can construct a new Hamiltonian $K$ in the form

$$K(y,Y,t;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} K_n(y,Y,t) \tag{64}$$

where to second-order

$$K_0 = H_0 \tag{65a}$$

$$K_1 = H_1 - DW_1/Dt \tag{65b}$$

$$K_2 = H_2 + L_1H_1 + L_1K_1 - DW_2/Dt \tag{65c}$$

$$DW_n/Dt = \partial W_n/\partial t - L_nH_0 \tag{65d}$$

The linear operator $L_n$ is called the Lie derivative; it is defined by

$$L_nf = (f;W_n) = f_y \cdot W_{nY} - f_Y \cdot W_{ny} \tag{65e}$$

The functions $W_1$ and $W_2$ are chosen to eliminate the short-period terms in $K_1$ and $K_2$.

Taking $y = (B_1^*,B_2^*)$, and $Y = (\alpha_1^*,\alpha_2^*)$ gives $K_1 = 0$ because $H_1$ contains short-period terms only. Then

$$K = \bar{\alpha}_1 - \bar{\alpha}_2 + 2e_1\bar{\alpha}_1^2 + 2e_2\bar{\alpha}_1\bar{\alpha}_2 + 2e_3\bar{\alpha}_2^2 - 4e_4\bar{\alpha}_1^{1/2}\bar{\alpha}^{3/2}\cos\gamma \tag{66}$$

where $\gamma = B_1^* - 3B_2^* + \phi$ and $\bar{\alpha}_i = \omega_i\alpha_i^*$. Since $\alpha_i^*$, and $B_1^*$ and $-B_2^*$ are canonical,

$$\begin{aligned} \dot{\alpha}_1^* &= -K_{B1^*}, \quad \dot{\alpha}_2^* = K_{B2^*} \\ \dot{B}_1^* &= K_{\alpha 1^*}, \quad \dot{B}_2^* = -K_{\alpha 2^*} \end{aligned} \tag{67}$$

Since $A_i = (2\omega_i\alpha_i)^{1/2}$, $B_1^* = \omega_1(t + \beta_1^*)$, $B_2^* = \omega_2(t - \beta_2^*)$, Eqs. (67) are identical with Eqs. (44–47). Thus, the results obtained by Lie transforms are in full agreement with those obtained using the method of multiple scales. The relationship between the constants $c_1$ and $c_2$ of Eqs. (51) and (52) and the Hamiltonian $K$ is found to be

$$c_2 = -3\omega_2K - c_1 - e_2c_1^2/4\omega_1 \tag{68}$$

## 5. Summary

Second-order expansions are presented for the motion of a particle near $L_4$ for $0.0125 \leq \mu \leq 0.015$ using the method of multiple scales[8,9] and the recent recurrence formulas of Refs. 6 and 7. The results obtained by these two methods are in full agreement with each other.

The linearized analysis[1,2] predicts for the above range of $\mu$ stable solutions having two modes of oscillation with amplitude independent frequencies $\omega_1$ and $\omega_2$ ($\omega_1 \approx 3\omega_2$). Moreover, it predicts that stable periodic orbits with either frequency are possible. On the other hand, the nonlinear theory shows that periodic orbits have amplitude-dependent frequencies. It predicts that short periodic orbits are unstable in the sense that any small disturbance would lead to a new aperiodic, though bounded, motion.

The family of long period orbits is given by a third-order algebraic curve $\Gamma$ in the $|A_2| - |A_1|$ plane. To each value of $\mu$ and $|A_1|$ corresponds only one value of $|A_2|$, while to each value of $\mu$ and $A_2$ corresponds either one or three values of $|A_1|$. If $\mu = \mu_3$, $\Gamma = \Gamma_3$ is given by the straight line $|A_2| = 0.181 |A_1|$. If $\mu < \mu_3$, $\Gamma$ consists of two branches that intersect the $|A_1|$ axis at a point $|\tilde{A}_{10}|$ given by Eq. (56). The two branches form a cusp at this point. As $\mu \to \mu_3$, the branch beyond $\tilde{A}_{10}$ tends to $\Gamma_3$ while the branch between the origin and $\tilde{A}_{10}$ collapses. If $\mu > \mu_3$, $\Gamma$ consists of only one branch that intersects the $|A_1|$ axis at the origin only, and $\Gamma$ tends to $\Gamma_3$ as $\mu \to \mu_3$. It should be mentioned that the present model, where long period orbits end by triplication on a short period orbit, is not valid outside the interval $\mu^{**} \leq \mu \leq \mu^*$. Above $\mu^*$, long period orbits end by duplication on a short period orbit, while below $\mu^{**}$, long period orbits end by quadruplication on a short period orbit. The values of $\mu^*$ and $\mu^{**}$ were pinned down by Deprit and co-workers.

At $\mu_3$, $L_4$ and $L_5$ are unstable, and all long periodic orbits near $L_4$ and $L_5$ are unstable. For each value of $\mu \neq \mu_3$ there exists a cut-off amplitude $|A_1|$ above which long period orbits are unstable while below which they are stable. For each value of $\mu$, the motion of a particle near $L_4$ or $L_5$ is stable or unstable depending on whether $|A_1|$ and $|A_2|$ correspond to a point below or above $\Gamma$.

## Appendix A

$$a_{10} = h_1(\omega_1), \quad a_{15} = h_1(\omega_2), \quad a_{11} = h_2(\omega_1)$$

$$a_{12} = h_2(\omega_2) \tag{A1}$$

$$h_{1,2}(\omega_i) = \frac{3^{1/2}}{48}\frac{k_i^2\eta}{\omega_i^2}[14m_i^2 + 9m_i - 22(\eta^2 \mp 4\omega_i^2)] \tag{A2}$$

$$m_i = \omega_i^2 + \tfrac{9}{4} \tag{A3}$$

$$b_{11} = h_3(\omega_1), \quad b_{12} = h_3(\omega_2) \tag{A4}$$

$$h_3(\omega_i) = \frac{3^{1/2}}{48}\frac{k_i^2}{\omega_i}(9m_i - 44\eta^2) \tag{A5}$$

$$a_{13},a_{14} = \frac{3^{1/2}}{48}\frac{k_1k_2\eta}{\omega_1\omega_2}[9(m_1 + m_2) - 44(\eta^2 \pm 4\omega_1\omega_2 + 28m_1m_2)] \tag{A6}$$

$$b_{13},b_{14} = \frac{3^{1/2}}{24}\frac{k_1k_2}{\omega_1\omega_2}[9(m_2\omega_1 \mp m_1\omega_2) - 44(\omega_1 \mp \omega_2)] \tag{A7}$$

$$a_{20} = g_1(\omega_1), \quad a_{25} = g_1(\omega_2), \quad a_{21} = g_2(\omega_1), a_{22} = g_2(\omega_2) \tag{A8}$$

$$g_{1,2}(\omega_i) = -\frac{3^{1/2}}{96}\frac{k_i}{\omega_i^2}[9m_i^2 - 88\eta^2m_i + 27(\eta^2 \mp 4\omega_i^2)] \tag{A9}$$

$$b_{21} = g_3(\omega_1), \quad b_{22} = g_3(\omega_2) \tag{A10}$$

$$g_3(\omega_i) = \frac{3^{1/2}}{24}\frac{k_i^2\eta}{\omega_i}(44m_i - 27) \tag{A11}$$

$$a_{23},a_{24} = -\frac{3^{1/2}}{48}\frac{k_1k_2}{\omega_1\omega_2}[9m_1m_2 - 44\eta(m_1 + m_2) + 27(\eta^2 \pm 4\omega_1\omega_2)] \tag{A12}$$

$$b_{23},b_{24} = \frac{3^{1/2}}{24}\frac{k_1k_2\eta}{\omega_1\omega_2}[44(m_2\omega_1 \mp m_1\omega_2) - 27(\omega_1 \mp \omega_2)] \tag{A13}$$

## Appendix B

If $b_{i0}$ and $b_{i5}$ ($i = 1$ and 2) are taken to be zero, the coefficients on the right-hand side are given by

$$c_{1j} = [2\lambda_j b_{2j} + \eta a_{2j} - (\lambda_j^2 + \tfrac{9}{4})\, a_{1j}]/\sigma_j \quad \text{(B1)}$$

$$s_{1j} = [\eta b_{2j} - 2\lambda_j a_{2j} - (\lambda_j^2 + \tfrac{9}{4}) b_{1j}]/\sigma_j \quad \text{(B2)}$$

$$c_{2j} = [\eta a_{1j} - 2\lambda_j b_{1j} - (\lambda_j^2 + \tfrac{3}{4}) a_{2j}]/\sigma_j \quad \text{(B3)}$$

$$s_{2j} = [2\lambda_j a_{1j} + \eta b_{1j} - (\lambda_j^2 + \tfrac{3}{4}) b_{2j}]/\sigma_j \quad \text{(B4)}$$

$$\sigma_j = (\lambda_j^2 + \tfrac{3}{4})(\lambda_j^2 + \tfrac{9}{4}) - (4\lambda_j^2 + \eta^2) \quad \text{(B5)}$$

where $j = 0,1,\ldots,5$ and

$$\lambda_0 = \lambda_5 = 0, \quad \lambda_1 = 2\omega_1, \quad \lambda_2 = 2\omega_2$$
$$\lambda_3 = \omega_1 + \omega_2, \quad \lambda_4 = \omega_1 - \omega_2 \quad \text{(B6)}$$

## Appendix C

$$P_{11} = 7.362A_1^3 + 108.8A_1A_2^2 - (28.48\cos\tilde{\gamma} - 104.1\sin\tilde{\gamma})A_2^3 + 1.518A_1' + 1.679\omega_1 A_1\beta_1' \quad \text{(C1)}$$

$$P_{12} = 2.643A_1^2A_2 + (15.46\cos\tilde{\gamma} - 69.57\sin\tilde{\gamma})A_1A_2^2 + 45.24A_2^3 + 2.505A_2' - 5.830\omega_2 A_2\beta_2' \quad \text{(C2)}$$

$$P_{21} = -6.514A_1^3 + 81.02A_1A_2^2 + (92.71\cos\tilde{\gamma} + 105.7\sin\tilde{\gamma})A_2^3 - 1.529A_1' + 3.561\omega_1 A_1\beta_1' \quad \text{(C3)}$$

$$P_{22} = -9.354A_1^2A_2 + (66.24\cos\tilde{\gamma} - 83.63\sin\tilde{\gamma})A_1A_2^2 - 54.41A_2^3 - 1.844A_2' - 9.917\omega_2 A_2\beta_2' \quad \text{(C4)}$$

$$Q_{11} = -10.72A_1^3 + 1.430A_1A_2^2 + (104.1\cos\tilde{\gamma} + 28.48\sin\tilde{\gamma})A_2^3 - 1.679A_1' + 1.579\omega_1 A_1\beta_1' \quad \text{(C5)}$$

$$Q_{12} = -16.51A_1^2A_2 + (69.57\cos\tilde{\gamma} + 15.46\sin\tilde{\gamma})A_1A_2^2 - 164.4A_2^3 - 5.830A_2' - 2.505\omega_2 A_2\beta_2' \quad \text{(C6)}$$

$$Q_{21} = -16.56A_1^3 - 126.7A_1A_2^2 + (105.7\cos\tilde{\gamma} - 92.71\sin\tilde{\gamma})A_2^3 - 3.516A_1' - 1.592\omega_1 A_1\beta_1' \quad \text{(C7)}$$

$$Q_{22} = -28.51A_1^2A_2 + (83.63\cos\tilde{\gamma} + 66.24\sin\tilde{\gamma})A_1A_2^2 - 276.8A_2^3 - 9.917A_2' + 1.844\omega_2 A_2\beta_2' \quad \text{(C8)}$$

$$\tilde{\gamma} = B_1 - 3B_2 = \omega_1\beta_1 + 3\omega_2\beta_2 = \omega_1(\beta_1 + \beta_2) \quad \text{(C9)}$$

where primes denote differentiation with respect to $T_2$.

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